Proving the trig equation

We've already talked a little bit about the reciprocal identities

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

but there are lots of other trig identities we can use to solve trig equations. In this section we're looking at the Pythagorean identities, which are

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

These are really helpful identities, because each one provides a relationship between two of the trig functions: sine and cosine, tangent and secant, and cotangent and cosecant.

These are used so often that you'll want to memorize them. But we can also derive them. Let's look at how to derive $\sin^2\theta + \cos^2\theta = 1$ so we have an idea about why it's true.

We need to remember that $\cos\theta$ and $\sin\theta$ are the x- and y-coordinates, respectively, of points on the unit circle. Since

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$$x^2 + y^2 = 1$$

for all points (x, y) on the unit circle, we can substitute $\cos\theta$ for x, and $\sin\theta$ for y, which gives

$$\cos^2\theta + \sin^2\theta = 1$$

This is the basic Pythagorean identity. We could write this identity in lots of other ways:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 = \sin^2\theta + \cos^2\theta$$

$$1 = \cos^2\theta + \sin^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta$$

The other two Pythagorean identities are derived from the basic Pythagorean identity. To get one of these other two identities, we'll first divide both sides of the basic identity by $\cos^2\theta$:

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

We'll simplify the second term on the left side.

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$$1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{1}{\cos^2\theta}$$

$$1 + (\tan\theta)^2 = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

By the reciprocal identity

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec^2\theta = \frac{1}{\cos^2\theta}$$

we get

$$1 + \tan^2\theta = \sec^2\theta$$

This is the second Pythagorean identity. We could write this identity in lots of other ways:

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec^2\theta = \tan^2\theta + 1$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$\sec^2\theta - 1 = \tan^2\theta$$

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To find the third Pythagorean identity, we'll divide the basic identity by $\sin^2\theta$ instead of $\cos^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \frac{1}{\sin^2\theta}$$

$$1 + (\cot\theta)^2 = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

This is the third Pythagorean identity. We could write this identity in lots of other ways:

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\csc^2\theta = \cot^2\theta + 1$$

$$\csc^2\theta = 1 + \cot^2\theta$$

$$\cot^2\theta = \csc^2\theta - 1$$

$$\csc^2\theta - 1 = \cot^2\theta$$

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When we use trig identities to prove a trig equation, we often start with one side of the equation and manipulate it by using algebra and one or more trig identities in order to get the expression on the other side of the equation.

There are lots of trigonometric identities. In the examples given in this lesson, we'll use only the reciprocal identities and the Pythagorean identities. We'll also use the definitions of $\tan\theta$ and $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example

Prove the trig equation.

$$\frac{\cot\theta}{\csc\theta} = \cos\theta$$

We can’t simplify the right side at all, so we’ll try to rewrite the left side to see if we can get it to equal $\cos\theta$. Since we know

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

we can rewrite the original equation as

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$$\frac{\frac{\cos\theta}{\sin\theta}}{\csc\theta} = \cos\theta$$

We know the reciprocal identity

$$\csc\theta = \frac{1}{\sin\theta}$$

so

$$\frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \cos\theta$$

$$\frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{1}\right) = \cos\theta$$

$$\frac{\cos\theta}{1}\left(\frac{1}{1}\right) = \cos\theta$$

$$\cos\theta = \cos\theta$$

Next, we'll prove an equation using a Pythagorean identity.

Example

Prove the trig equation.

$$\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = 1 - 2\cos^2\theta$$

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We'll start with the left side, and try to show that it's equal to the right side. We know the quotient identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

so

$$\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = 1 - 2\cos^2\theta$$

$$\frac{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}} = 1 - 2\cos^2\theta$$

Find common denominators in both the numerator and denominator, then simplify them.

$$\frac{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right) - \frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right) + \frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)} = 1 - 2\cos^2\theta$$

$$\frac{\frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{\cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}} = 1 - 2\cos^2\theta$$

$$\frac{\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}} = 1 - 2\cos^2\theta$$

$$\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}\left(\frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta}\right) = 1 - 2\cos^2\theta$$

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$$\frac{\sin^2\theta - \cos^2\theta}{1}\left(\frac{1}{\sin^2\theta + \cos^2\theta}\right) = 1 - 2\cos^2\theta$$

$$\frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta} = 1 - 2\cos^2\theta$$

Since we know the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we get

$$\frac{\sin^2\theta - \cos^2\theta}{1} = 1 - 2\cos^2\theta$$

$$\sin^2\theta - \cos^2\theta = 1 - 2\cos^2\theta$$

Using one of the other forms of the basic Pythagorean identity, $\sin^2\theta = 1 - \cos^2\theta$, we'll make a substitution for $\sin^2\theta$.

$$\left(1 - \cos^2\theta\right) - \cos^2\theta = 1 - 2\cos^2\theta$$

$$1 - \cos^2\theta - \cos^2\theta = 1 - 2\cos^2\theta$$

$$1 - 2\cos^2\theta = 1 - 2\cos^2\theta$$

Sometimes it's easier or more convenient to work on the expressions on the two sides of the equation separately (one after the other), and prove that they're both equal to a third expression.

Example

Prove the trig equation.

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$$\csc\theta + \cot\theta = \frac{\sin\theta}{1-\cos\theta}$$

We'll start by working on the left side. We know the reciprocal and quotient identities

$$\csc\theta = \frac{1}{\sin\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

so we'll rewrite the left side to get

$$\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$$

We've got a common denominator on the left side, so

$$\frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$$

Now we'll work on the right side. We'll multiply both the numerator and denominator by the conjugate of the denominator, $1+\cos\theta$. Remember that the conjugate of a two-term expression is the same two terms, but with the sign in the middle flipped. So the conjugate of $1-\cos\theta$ is $1+\cos\theta$.

$$\frac{1+\cos\theta}{\sin\theta} = \left(\frac{\sin\theta}{1-\cos\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$

Use the rewritten form $1-\cos^2\theta = \sin^2\theta$ of the basic Pythagorean identity to replace the denominator on the right side.

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$$\frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta}$$

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